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A quiver of many runaways

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Abstract: We study the quantum corrections to the moduli space of the quiver gauge theory corresponding to regular and fractional D3-branes at the dP_1 singularity. We find that besides the known runaway behavior at the lowest step of the duality cascade, there is a runaway direction along a mesonic branch at every higher step of the cascade. Moreover, the algebra of the chiral operators which obtain the large expectation values is such that we reproduce Altmann's first order deformation of the dP_1 cone.

Keywords: Supersymmetry Breaking, Differential and Algebraic Geometry, Gauge-gravity correspondence, Supersymmetric gauge theory.

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1. Introduction

One of the most important advances in the study of the holographic duality between gauge theories and string backgrounds was the generalization of the AdS/CFT correspondence from D3-branes in flat space [1] to D3-branes probing Calabi-Yau (CY) singularities [2 – 4]. When the latter singularities are toric, a rigorous correspondence between the algebraic-geometric properties of the singularity and the resulting quiver gauge theory has been established over the years. In particular, it is very beautiful to see how the complex equations characterizing the geometry arise by solving for the classical moduli space of the $\mathcal{N} = 1$ superconformal quiver gauge theory [5, 6].

In the case of the conifold singularity, it is known that there is a complex deformation which leads to a smooth CY geometry, namely the deformed conifold. The latter geometry

also arises as a moduli space of a quiver gauge theory [7], in which however we have to depart from conformality and introduce a non-trivial renormalization group (RG) flow. From the stringy point of view, this is triggered by the presence of fractional branes. It is argued that in the deep IR one ends up with a confining SYM gauge theory, and the deformation parameter of the geometry is related to the gaugino condensate. In fact, as soon as some fractional branes are included, the quantum moduli space of the quiver gauge theory separates in several branches, each one representing a number of regular branes probing the deformed conifold [8]. More specifically, the branches associated to wandering regular branes are mesonic branches from the gauge theory point of view, while the empty, smooth geometry is associated to the baryonic branch of the quiver gauge theory.

However, this behavior is not the typical one. In other geometries, fractional branes trigger an RG flow which, after a cascade of Seiberg dualities, does not end in confining vacua, but rather in a theory which breaks supersymmetry with a runaway behavior [9 – 12], similarly to massless SQCD with $N_f < N_c$ [13]. In the following, we will consider the complex cone over the first del Pezzo surface, or in short dP_1, as the representative of such geometries. Its quiver gauge theory was derived in [14]. The dP_1 is known to have an obstructed complex deformation [15], that is, a complex deformation at first order which however has to vanish at second order for consistency.

In the present paper, we consider in detail the possible solutions to the quantum F-term equations of the dP_1 quiver gauge theory. The classical moduli space is consistently lifted everywhere, however we show that the F-terms can be satisfied at infinity in field space on every mesonic branch, signalling a runaway behavior. There are as many runaway directions as there are steps in the duality cascade. Moreover, along the runaway directions, the gauge invariants reproduce the equations of the singularity deformed at first order. In other words, the regular D3-branes are pushed to infinity, but as they run away, they are probing a geometry corresponding exactly to the first order deformation of Altmann. Note that the latter is not CY (and hence the background not supersymmetric) at quadratic order in the deformation, while of course the gauge theory breaks supersymmetry because of the non-vanishing F-terms. We hence observe a nice check of the gauge/string correspondence which goes beyond the usual, protected, supersymmetric vacua but rather extends to situations with only asymptotic supersymmetry.

The plan of the paper is the following. In section 2 we consider the classical moduli space of the dP_1 quiver gauge theory, paying attention to mesonic and baryonic branches and their being decoupled. In section 3 we derive the quantum corrections, identifying all the runaway directions and making the relation with the obstructed deformation of the geometry. Some discussion is found in section 4. In appendix A, we apply the same analysis as in the main text to study the various branches of the moduli space of the conifold gauge theory, in order to "normalize" our approach in a well-known example. In appendix B, we make a similar analysis of the runaway mesonic branch for the quiver gauge theory corresponding to supersymmetry breaking fractional branes at the dP_2 singularity, for which we also derive the obstructed deformation.

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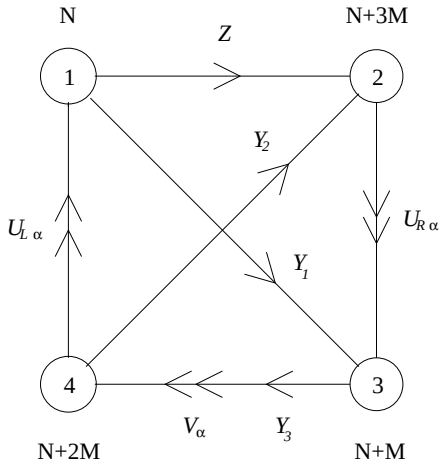

Figure 1: The dP_1 quiver for N regular and M fractional branes.

2. The classical moduli space of the dP_1 quiver gauge theory

The quiver gauge theory corresponding to D3-branes probing a dP_1 singularity has gauge group $\mathrm{SU}(N) \times \mathrm{SU}(N+3M) \times \mathrm{SU}(N+M) \times \mathrm{SU}(N+2M)$ and matter fields which can be read out from the diagram reproduced in figure 1.

The superpotential is

$$W_{\text{tree}} = h\mathrm{Tr}(\epsilon^{\alpha\beta} Y_3 U_{L\alpha} Z U_{R\beta} - \epsilon^{\alpha\beta} V_\alpha Y_2 U_{R\beta} + \epsilon^{\alpha\beta} V_\alpha U_{L\beta} Y_1), \tag{2.1}$$

where we choose to trace over the node 3 gauge group indices. Remark that we only have a diagonal SU(2) flavor symmetry. For later convenience, we already introduce variables which are suitable for describing objects which are gauge invariant with respect to node 2, the one with highest rank

$$\mathcal{M}_\alpha = Z U_{R\alpha}, \qquad \text{and} \qquad \mathcal{N}_\alpha = Y_2 U_{R\alpha}. \tag{2.2}$$

The classical F-term equations derived from extremizing (2.1) are

$$\epsilon^{\alpha\beta} U_{R\alpha} Y_3 U_{L\beta} = 0, \tag{2.3}$$
$$\epsilon^{\alpha\beta} V_\alpha U_{L\beta} = 0, \tag{2.4}$$
$$\epsilon^{\alpha\beta} U_{R\alpha} V_\beta = 0, \tag{2.5}$$
$$\epsilon^{\alpha\beta} U_{L\alpha} Z U_{R\beta} = 0, \tag{2.6}$$
$$V_\alpha Y_2 = Y_3 U_{L\alpha} Z, \tag{2.7}$$
$$Y_1 V_\alpha = Z U_{R\alpha} Y_3, \tag{2.8}$$
$$Y_2 U_{R\alpha} = U_{L\alpha} Y_1. \tag{2.9}$$

2.1 The mesonic branch

For generic N, we can build basic "loops" which consist of products of 3 or 4 bifundamentals such that the resulting object has both indices in one gauge group [9]. We have a total of 12

loops going through nodes 1-2-3-4, 6 loops going through nodes 1-3-4 and 6 loops through nodes 2-3-4. It would thus seem that if we base ourselves on nodes 1 or 2 we will see less loops and possibly a reduced moduli space. However this is not true because the F-terms reduce the number of independent loops to 9, and eventually equate the eigenvalues of the loops based on different nodes. We briefly sketch below how this happens. See also appendix A where the same approach is applied in all details to the conifold gauge theory.

Let us for definiteness base ourselves on node 3. We immediately see that the loop matrices will be distinguished by the number of SU(2) indices that they carry: one, two or three. Using the F-terms, we have respectively

$$Y_3 U_{L\alpha} Y_1 = Y_3 Y_2 U_{R\alpha}, \tag{2.10}$$

$$V_\alpha U_{L\beta} Y_1 = V_\alpha Y_2 U_{R\beta} = Y_3 U_{L\alpha} Z U_{R\beta} = Y_3 U_{L(\alpha} Z U_{R\beta)}, \tag{2.11}$$

$$V_\alpha U_{L\beta} Z U_{R\gamma} = V_{(\alpha} U_{L\beta} Z U_{R\gamma)}. \tag{2.12}$$

In particular, we see that all the SU(2) indices are symmetrized because of the first four F-term relations. We thus end up indeed with 9 elementary loops, which we can name as follows, using the gauge invariants of node 2 introduced in (2.2)

$$\begin{aligned} a_1 &= Y_3 \mathcal{N}_1, \quad & b_1 &= Y_3 U_{L1} \mathcal{M}_1, \quad & c_1 &= V_1 U_{L1} \mathcal{M}_1, \\ a_2 &= Y_3 \mathcal{N}_2, \quad & b_2 &= Y_3 U_{L1} \mathcal{M}_2, \quad & c_2 &= V_1 U_{L1} \mathcal{M}_2, \\ & & b_3 &= Y_3 U_{L2} \mathcal{M}_2, \quad & c_3 &= V_2 U_{L1} \mathcal{M}_2, \\ & & & & c_4 &= V_2 U_{L2} \mathcal{M}_2. \end{aligned} \tag{2.13}$$

These 9 matrices commute, as one can easily check, so we can diagonalize them all. Moreover, they are not independent. There are 20 quadratic relations between them, defining the complex cone over the first del Pezzo as a 3 dimensional affine variety in $\mathbb{C}^9$ [9]:

$$\begin{array}{cccc} a_1 b_2 = a_2 b_1 & a_1 b_3 = a_2 b_2 & b_2^2 = b_1 b_3 & b_2^2 = a_1 c_3 \\ b_2^2 = a_2 c_2 & b_1^2 = a_1 c_1 & b_3^2 = a_2 c_4 & a_1 c_2 = b_1 b_2 \\ a_1 c_4 = b_2 b_3 & a_2 c_1 = b_1 b_2 & a_2 c_3 = b_2 b_3 & b_1 c_2 = b_2 c_1 \\ b_1 c_3 = b_2 c_2 & b_1 c_4 = b_2 c_3 & b_2 c_3 = b_3 c_2 & b_2 c_4 = b_3 c_3 \\ b_2 c_2 = b_3 c_1 & c_1 c_4 = c_2 c_3 & c_2^2 = c_1 c_3 & c_3^2 = c_2 c_4 \end{array} \tag{2.14}$$

As complicated as they look, all the above relations can easily be seen to arise just by considering that all quadratic objects with the same SU(2) indices must coincide and be symmetrized.

Exactly the same conclusion can be reached considering loops on any one of the other three nodes. Note that because of their definitions (and because of the above equations), all loops are eventually matrices of rank N even when they are based on nodes of higher rank. Moreover, the most generic situation is when all non vanishing eigenvalues are, say, in the upper-left corner.

In conclusion, the moduli space is a N-symmetric product of the CY affine variety.

We write now an explicit parametrization for the classical moduli space, which means solving for the D-terms and F-terms simultaneously. Our approach has been of course to first solve the F-flatness conditions and then worry about the D-terms.

We adopt the solution of the F-terms given above by 9 mutually commuting matrices (2.13) at every node. Choose for instance the loops

$$a_1^{(3)} = Y_3 Y_2 U_{R1}, \qquad a_1^{(4)} = Y_2 U_{R1} Y_3, \tag{2.15}$$

based at node 3 and 4, respectively. Our loops obviously cannot have rank larger than N. Now use gauge freedom from gauge groups 3 and 4 to gauge fix

$$(a_1^{(3)})^i_j = (a_1^{(3)})^i \delta^i_j, \quad (a_1^{(4)})^i_j = (a_1^{(4)})^i \delta^i_j, \qquad \text{for} \quad i,j \leq N, \tag{2.16}$$

zero otherwise. Now, $a_1^{(3)} Y_3 = Y_3 a_1^{(4)}$ implies that, for generic vevs, Y_3 is diagonal in the upper-left $N \times N$ corner. One has (without summation):

$$(a_1^{(3)})^i Y^i_{3j} = Y^i_{3j} (a_1^{(4)})^j \quad \Rightarrow \quad Y^i_{3j} = 0 \quad \text{if} \quad i \neq j \quad \text{and} \quad (a_1^{(3)})^i = (a_1^{(4)})^i \tag{2.17}$$

for $i, j \leq N$, while for $p > N$,

$$(a_1^{(3)})^i Y^i_{3p} = 0, \qquad Y^p_{3i} (a_1^{(4)})^i = 0. \tag{2.18}$$

Note that the components Y^p_{3q} for $p, q > N$ remain undetermined.

By similar arguments, we can arrive at the conclusion that all sets of basic loops based on different nodes actually share the same eigenvalues, and all elementary fields have a diagonal upper-left $N \times N$ part and an undetermined lower-right piece whose dimension is $M \times 2M$ for Y_3, V_α, $2M \times 3M$ for Y_2 and $3M \times M$ for $U_{R\alpha}$. To summarize, we have thus shown that all the bifundamental fields must have the form

$$X = \begin{pmatrix} X^D_{N\times N} & 0 \\ 0 & \tilde{X} \end{pmatrix}. \tag{2.19}$$

We assume for the moment that $\tilde{X} = 0$ for all the fields, so that all the vevs are diagonal. We list the additional constraints from the D-equations:

$$|Z_i|^2 + |Y_{1i}|^2 - |U_{L1i}|^2 - |U_{L2i}|^2 = 0, \tag{2.20}$$

$$|U_{R1i}|^2 + |U_{R2i}|^2 - |Z_i|^2 - |Y_{2i}|^2 = 0, \tag{2.21}$$

$$|V_{1i}|^2 + |V_{2i}|^2 + |Y_{3i}|^2 - |Y_{1i}|^2 - |U_{R1i}|^2 - |U_{R2i}|^2 = 0, \tag{2.22}$$

$$|Y_{2i}|^2 + |U_{L1i}|^2 + |U_{L2i}|^2 - |V_{1i}|^2 - |V_{2i}|^2 - |Y_{3i}|^2 = 0, \tag{2.23}$$

where $i = 1, \ldots N$ runs over the upper-left diagonal blocks. The pattern of higgsing of the gauge group is

$$\begin{aligned} G &= \mathrm{SU}(N) \times \mathrm{SU}(N+3M) \times \mathrm{SU}(N+M) \times \mathrm{SU}(N+2M) \\ &\supset \mathrm{SU}(N)_{\mathrm{diag}} \times \mathrm{SU}(3M) \times \mathrm{SU}(M) \times \mathrm{SU}(2M) \\ &\supset \mathrm{U}(1)^{N-1} \times \mathrm{SU}(3M) \times \mathrm{SU}(M) \times \mathrm{SU}(2M), \end{aligned} \tag{2.24}$$

where the non-abelian part is the dP_1 quiver for $N = 0$ (i.e. the triangle quiver), while the $\mathrm{U}(1)$'s are diagonal combinations of the Cartan subalgebras of the four nodes' $\mathrm{SU}(N)$ subgroups.

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2.2 The baryonic branches

Let us now consider the special case $N = M$, which we take as a case study of the more general situation $N = kM$. In this case we can define baryonic gauge invariants for the second node with SU($4M$) gauge group, since it has effectively $N_f = 4M$.[1]

The mesonic gauge invariants of node two are $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ as defined in (2.2). They are respectively pairs of $M \times 2M$ and $3M \times 2M$ matrices. We can thus define a $4M \times 4M$ mesonic matrix as

$$\widetilde{\mathcal{M}} \equiv \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_2 \\ \mathcal{N}_1 & \mathcal{N}_2 \end{pmatrix}. \tag{2.25}$$

More generally, we can define the same matrix also when $N \neq M$ and it will be $(2N + 2M) \times (2N + 2M)$. In the mesonic branch considered above, it is clear that the matrices $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ are non zero only in the upper left rank N part. Hence, $\widetilde{\mathcal{M}}$ will be of maximal rank $2N$ and $\det \widetilde{\mathcal{M}} = 0$. Actually, from the classical F-terms (2.9) we see that the matrices $\mathcal{N}_\alpha$ are each of maximal rank N (since there is a summation over SU(N) indices on the r.h.s.) so that, when $N = M$ we necessarily have $\det \widetilde{\mathcal{M}} = 0$ in any SUSY vacuum. This is going to play an important role in the subsequent analysis.

In the $N = M$ case we can define two baryonic invariants,

$$\mathcal{B} \propto (Y_2)^{3M} Z^M \equiv \det \begin{pmatrix} Z \\ Y_2 \end{pmatrix}, \tag{2.26}$$

$$\bar{\mathcal{B}} \propto (U_{R1} U_{R2})^{2M} \equiv \det \begin{pmatrix} U_{R1} & U_{R2} \end{pmatrix}, \tag{2.27}$$

where both matrices entering the definitions are $4M \times 4M$. Again, it is easy to see that on the mesonic branch $\mathcal{B}, \bar{\mathcal{B}} = 0$ because of the non-maximal ranks of the matrices involved.

We now ask whether it is possible to have regions or branches of the moduli space where the baryonic invariants are turned on. We see that we can use the F-term (2.7) in order to form gauge invariants involving $\mathcal{B}$. We get the equations

$$V_\alpha \mathcal{B} = 0 = Y_3 U_{R\alpha} \mathcal{B}. \tag{2.28}$$

This means that if $\mathcal{B} \neq 0$, then $V_\alpha = Y_3 U_{R\alpha} = 0$ identically. It is easy to show that this in turn implies that all basic loops (2.13) vanish. Similarly, from the F-terms (2.3) and (2.5) we obtain

$$V_\alpha \bar{\mathcal{B}} = 0 = Y_3 U_{R\alpha} \bar{\mathcal{B}}, \tag{2.29}$$

with the same conclusion of vanishing loops. Moreover, if $\mathcal{B} \neq 0$ then $\bar{\mathcal{B}}$ has to vanish and vice-versa because $\mathcal{B}\bar{\mathcal{B}} = \det \widetilde{\mathcal{M}} = 0$.

Thus we see that additionally to the mesonic branch, which consists of M symmetrized copies of the complex cone over dP_1, we have two one-complex dimensional baryonic branches. All of the branches of the moduli space meet at the origin.

[1] Actually, baryonic gauge invariants can generally be defined, for any node, when $N = kM$. However only when we have $N_f = N_c$ for one node do the baryons become elementary effective fields at the quantum level.

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In order to see what is the left over gauge group on the baryonic branches, we have to solve for the D-terms.[2] Because the loops are all zero, we are more constrained than on the mesonic branch and the elementary fields will have VEVs proportional to the identity. More explicitely, when $\mathcal{B} \neq 0$ we turn on only the Z and Y_2 fields. It turns out that we have to take the left $M \times M$ part of Z and the right $3M \times 3M$ part of Y_2 proportional to the identity, with the same constant of proportionality. The gauge group is broken according to the following pattern:

$$\begin{aligned} G &= \mathrm{SU}(M) \times \mathrm{SU}(4M) \times \mathrm{SU}(2M) \times \mathrm{SU}(3M) \\ &\supset \mathrm{SU}(M) \times (\mathrm{SU}(M) \times \mathrm{SU}(3M)) \times \mathrm{SU}(2M) \times \mathrm{SU}(3M) \\ &\supset \mathrm{SU}(M)_{\mathrm{diag}} \times \mathrm{SU}(3M)_{\mathrm{diag}} \times \mathrm{SU}(2M). \end{aligned} \tag{2.30}$$

Thus, we get the triangular quiver, and the matter content can be checked to be the expected one by standard higgsing arguments. Note that on the baryonic branch we do not have a $\mathrm{U}(1)^{N-1}$ factor.

Similarly, on the $\bar{\mathcal{B}} \neq 0$ branch we have that U_{R1} and U_{R2} have respectively their upper and lower $2M \times 2M$ parts proportional to the identity because of the D-terms. The gauge group is broken according to

$$\begin{aligned} G &= \mathrm{SU}(M) \times \mathrm{SU}(4M) \times \mathrm{SU}(2M) \times \mathrm{SU}(3M) \\ &\supset \mathrm{SU}(M) \times (\mathrm{SU}(2M) \times \mathrm{SU}(2M)) \times \mathrm{SU}(2M) \times \mathrm{SU}(3M) \\ &\supset \mathrm{SU}(M) \times \mathrm{SU}(2M)_{\mathrm{diag}} \times \mathrm{SU}(3M), \end{aligned} \tag{2.31}$$

again obtaining the same theory, albeit embedded in a different way in the original gauge group.

Note that the fact that at any point of the various branches of the moduli space we still have a non-trivial gauge theory, namely the triangular quiver, means that each one of this points actually corresponds to a moduli space of its own. In other words, every point of the moduli space discussed here is itself a moduli space, which is the one discussed in detail in [12, 16].

3. The quantum corrections to the dP_1 moduli space

Here we wish to study how the classical picture is modified by quantum corrections. The story in the $N = kM$ case is by now well-known [9 – 11]. The gauge theory is non conformal and is believed to undergo a non-trivial RG flow which takes the form of a cascade of Seiberg dualities. The latter effectively reduce the ranks at every node by M at every step.[3] At the last step, one usually goes to the (quantum) baryonic branch and ends up with the triangular quiver, which is runaway as we will rederive later (see [12, 16] for a discussion on how one might stop this runaway behavior).

[2] Note that in this case the r.h.s. of the D-equations (2.20)–(2.23), i.e. the trace part, is non vanishing.

[3] That this RG flow has to be the one described by a gravity dual such as the ones in [17, 18] has been argued in [19].

In the language of the previous section, the above result can be stated by saying that the baryonic branch of the second node becomes runaway because of quantum corrections coming from another node (the fourth). Here we wish to address the question of what becomes of the mesonic branch of the second node. Because the low energy gauge group is still the triangular quiver, we also expect a runaway behaviour, but since the embedding of the gauge group is different the runaway will be driven by different quantum effects. Also, from the dual stringy perspective, on the mesonic branch we have regular D3-branes around and the question is whether they will feel a potential, or what space they will seem to be probing.

In the following, we start by considering the case $N = M$ which hopefully captures most of the physics we want to discuss. We will turn later to the more general case $N \neq M$.

3.1 Runaway on the baryonic branch

The effective superpotential for $N = M$ is

$$W = h\mathrm{Tr}(\epsilon^{\alpha\beta} Y_3 U_{L\alpha} \mathcal{M}_\beta - \epsilon^{\alpha\beta} V_\alpha \mathcal{N}_\beta + \epsilon^{\alpha\beta} V_\alpha U_{L\beta} Y_1) + L(\det \widetilde{\mathcal{M}} - \mathcal{B}\bar{\mathcal{B}} - \Lambda_2^{8M}), \tag{3.1}$$

where L is a superfield Lagrange multiplier.

The F-terms are the following

$$\det \widetilde{\mathcal{M}} - \mathcal{B}\bar{\mathcal{B}} = \Lambda_2^{8M}, \tag{3.2}$$

$$L\mathcal{B} = 0 = L\bar{\mathcal{B}}, \tag{3.3}$$

$$\epsilon^{\alpha\beta} Y_3 U_{L\alpha} = -L \frac{\partial \det \widetilde{\mathcal{M}}}{\partial \mathcal{M}_\beta}, \tag{3.4}$$

$$\epsilon^{\alpha\beta} V_\alpha = L \frac{\partial \det \widetilde{\mathcal{M}}}{\partial \mathcal{N}_\beta}, \tag{3.5}$$

$$\epsilon^{\alpha\beta} U_{L\alpha} \mathcal{M}_\beta = 0, \tag{3.6}$$

$$\epsilon^{\alpha\beta} V_\alpha U_{L\beta} = 0, \tag{3.7}$$

$$\mathcal{M}_\alpha Y_3 = Y_1 V_\alpha, \tag{3.8}$$

$$\mathcal{N}_\alpha = U_{L\alpha} Y_1. \tag{3.9}$$

As in the classical case, if we want to satisfy (3.9), then the matrix $\widetilde{\mathcal{M}}$ is not of maximal rank and $\det \widetilde{\mathcal{M}} = 0$. We are then automatically on the baryonic branch: the constraint (3.2) forces the baryons $\mathcal{B}, \bar{\mathcal{B}}$ to have non zero VEVs (at the quantum level they must be both non vanishing), which in turn implies $L = 0$ from (3.3). Then, (3.4) and (3.5) imply that V_α and $Y_3 U_{L\alpha}$ are zero, which eventually means that all the loop variables are zero. We are definitely on the baryonic branch which, as far as the dynamics of node two is concerned, is still supersymmetric. At this stage, note that the mesonic branch has no chance of appearing because non-vanishing loops would need $L \neq 0$ which would mean vanishing baryons and $\det \widetilde{\mathcal{M}} = \Lambda_2^{8M}$, contradicting one of the F-terms.

So, we see that quantum effects at node two lift the mesonic branch but not the baryonic one (which is the smooth merger of the two classical baryonic branches).

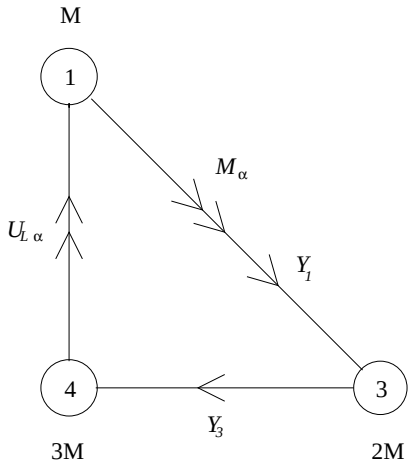

Figure 2: The last step of the cascade of the dP_1 quiver for M fractional branes.

For the sake of completeness, we reproduce here the well-known result that the baryonic branch is also eventually lifted by quantum corrections. Indeed, on the baryonic branch we are left with a triangular quiver with gauge group $\mathrm{SU}(M)\times\mathrm{SU}(2M)\times\mathrm{SU}(3M)$ and matter represented by $U_{L\alpha}$, Y_1, Y_3 and $\mathcal{M}_\alpha$ (V_α and $\mathcal{N}_\alpha$ have been integrated out because they appear quadratically in (3.1)). The quiver is represented in figure 2. The matter content is such that node four with gauge group $\mathrm{SU}(3M)$ has $N_f = 2M < N_c$ flavors. Hence, an ADS-like effective superpotential will be generated for its mesons $X_\alpha = Y_3 U_{L\alpha}$

$$W_{\text{eff}} = h\mathrm{Tr}\epsilon^{\alpha\beta}X_\alpha\mathcal{M}_\beta + M\left(\frac{\Lambda_4^{7M}}{\det\tilde{X}}\right)^{\frac{1}{M}}, \tag{3.10}$$

where we have defined the $2M\times 2M$ matrix $\tilde{X}\equiv(X_1\,X_2)$ and Λ_4 is the dynamical scale of node four. It is clear that the F-terms will set X_α to zero while sending $\mathcal{M}_\alpha$ to infinity. This is the runaway direction at the last step of the cascade. In the following, we want to see if there are other, disconnected runaway directions corresponding to the other branches of the classical moduli space.

3.2 Runaway on the mesonic branch

For simplicity, we consider here solutions to the F-terms (3.2)–(3.9) in the special case $N = M = 1$. We force being on the mesonic branch by requiring that $L \neq 0$ so that $\mathcal{B}, \bar{\mathcal{B}} = 0$ and

$$\det\widetilde{\mathcal{M}} = \Lambda^8. \tag{3.11}$$

We immediately see that, if we are to find a solution to the F-terms, it will be runaway, because the above condition conflicts with the rank condition following from (3.9). Hence, both equations will be satisfied only if some elements of $\widetilde{\mathcal{M}}$ go to infinity as others go to zero. The non-trivial task is to find a scaling for all the fields appearing above such that all F-terms go to zero while loop variables remain non zero. The Lagrange multiplier L

should also be large enough in order for this branch to be really disconnected from the baryonic one, as it is the case classically.

All fields will thus have a non zero VEV assigned to their upper-left component, which is the one entering in the loop variables. Additionally, at least the fields $\mathcal{N}_\alpha$ will have to have some non-zero component in the lower-right part. We will see that as a consequence also V_α will need to have such a component. We thus take

$$\mathcal{M}_\alpha = \begin{pmatrix} m_\alpha & 0 \end{pmatrix}, \qquad \mathcal{N}_\alpha = \begin{pmatrix} n_\alpha & 0 \\ 0 & \epsilon_\alpha \\ 0 & \delta_\alpha \end{pmatrix}, \tag{3.12}$$

$$Y_3 = \begin{pmatrix} y_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad V_\alpha = \begin{pmatrix} v_\alpha & 0 & 0 \\ 0 & w_\alpha & x_\alpha \end{pmatrix}, \tag{3.13}$$

$$Y_1 = \begin{pmatrix} y_1 & 0 \end{pmatrix}, \qquad U_{L\alpha} = \begin{pmatrix} u_\alpha \\ 0 \\ 0 \end{pmatrix}. \tag{3.14}$$

It is convenient to rewrite the effective superpotential (3.1) in terms of the above ansatz

$$W = y_3 u_\alpha m^\alpha - v_\alpha n^\alpha - w_\alpha \epsilon^\alpha - x_\alpha \delta^\alpha + v_\alpha u^\alpha y_1 - L(m_\alpha n^\alpha \epsilon_\beta \delta^\beta + \Lambda^8), \tag{3.15}$$

where all fields appearing are no longer matrices, and e.g. $m^\alpha \equiv \epsilon^{\alpha\beta} m_\beta$.

The F-terms simply read

$$u_\alpha m^\alpha = 0, \tag{3.16}$$
$$v_\alpha u^\alpha = 0, \tag{3.17}$$
$$y_3 m_\alpha = v_\alpha y_1, \tag{3.18}$$
$$n_\alpha = u_\alpha y_1, \tag{3.19}$$
$$\epsilon_\alpha = 0, \tag{3.20}$$
$$\delta_\alpha = 0, \tag{3.21}$$
$$y_3 u_\alpha = -L n_\alpha \epsilon_\beta \delta^\beta, \tag{3.22}$$
$$v_\alpha = -L m_\alpha \epsilon_\beta \delta^\beta, \tag{3.23}$$
$$w_\alpha = L \delta_\alpha m_\beta n^\beta, \tag{3.24}$$
$$x_\alpha = -L \epsilon_\alpha m_\beta n^\beta. \tag{3.25}$$

The F-terms setting ϵ_α and δ_α to zero are clearly the ones violating the condition $\det \widetilde{\mathcal{M}} = \Lambda^8$. The constraint $m_\alpha n^\alpha \epsilon_\beta \delta^\beta = -\Lambda^8$ will thus send $m_\alpha n^\alpha$ to infinity. Actually we will see that this scaling to infinity is subdominant with respect to the one of m_α and n_α. For concreteness, let us take $\epsilon_1 = \delta_2 = 0$ and $\epsilon_2 = \delta_1 = \epsilon$. As an immediate consequence, $x_1 = w_2 = 0$. Moreover,

$$m_\alpha n^\alpha = \frac{\Lambda^8}{\epsilon^2}, \tag{3.26}$$

so that

$$y_3 u_\alpha = L n_\alpha \epsilon^2, \quad v_\alpha = L m_\alpha \epsilon^2, \quad w_1 = L \frac{\Lambda^8}{\epsilon}, \quad x_2 = -L \frac{\Lambda^8}{\epsilon}. \tag{3.27}$$

Comparing with other F-terms, we see that $y_1 \propto y_3$ and $v_\alpha \propto m_\alpha$. Analyzing the scaling of the basic loops (2.13), we see that it is consistent to take the same scaling for y, m, v and u.[4] In this way, all loops will scale in the same way and modding out by the (possibly infinite) common factor we would obtain finite equations. We thus take

$$y_1 = y_3 = y, \qquad v_\alpha = m_\alpha. \tag{3.28}$$

We see that this implies

$$L = \frac{1}{\epsilon^2}, \qquad w_1 = \frac{\Lambda^8}{\epsilon^3}, \qquad x_2 = -\frac{\Lambda^8}{\epsilon^3}. \tag{3.29}$$

The Lagrange multiplier L goes to infinity, meaning that the mesonic branch analyzed here is effectively very far from the baryonic branch described previously. Note that the scaling to infinity behavior of w_1 and x_2 is related to the (additional) runaway behavior of the left-over triangle quiver at any point of the mesonic branch. Indeed, in the present case, the SU($3M$) node at low energies is embedded also in the SU($4M$) of the original quiver, and thus the latter's dynamical scale is also responsible for this "secondary" runaway behavior.

The most obvious way to satisfy the F-terms $m_\alpha u^\alpha = 0$ would be to take $m_\alpha = u_\alpha$. But that would contradict the constraint. Hence, we must add a subdominant piece as $m_\alpha = u_\alpha + m'_\alpha$, so that

$$m'_\alpha u^\alpha \to 0 \qquad \text{but} \qquad y m'_\alpha u^\alpha = \frac{\Lambda^8}{\epsilon^2}. \tag{3.30}$$

At this stage, there is some arbitrariness in the way we choose the scaling to zero of $m'_\alpha u^\alpha$.

For definiteness, we choose all the non vanishing F-terms to scale in the same way. Hence we take

$$m'_\alpha u^\alpha = \mathcal{O}(\epsilon). \tag{3.31}$$

This implies the following scaling for y

$$y = \mathcal{O}(\epsilon^{-3}). \tag{3.32}$$

As stated previously, we also take u_α to scale in the same way, $u_\alpha = \mathcal{O}(\epsilon^{-3})$. As a consequence, $m'_\alpha = \mathcal{O}(\epsilon^4)$. We can see that all the F-terms, and the constraint, are satisfied as $\epsilon \to 0$.[5] All the loops have a dominant piece which scales as $\mathcal{O}(\epsilon^{-9})$. They can actually all be expressed in terms of 3 variables, y, u_1 and u_2, so that

$$a_\alpha = y^2 u_\alpha, \quad b_{\alpha\beta} = y u_\alpha u_\beta, \qquad c_{\alpha\beta\gamma} = u_\alpha u_\beta u_\gamma. \tag{3.33}$$

This just reproduces the fact that, away from the singularity, the space probed by the D3-branes is locally $\mathbb{C}^3$ and thus the 20 equations defining the complex cone can be solved in terms of three complex variables. Note that we alternatively call $b_{11} \equiv b_1, b_{12} \equiv b_2, b_{22} \equiv b_3$ and similarly for the cs.

[4] Actually, the D-flatness conditions will be satisfied only if we take the elementary fields to scale as above.

[5] We are making the reasonable assumption that for large VEVs, the Kähler potential is close to being the classical canonical one. Hence also the vacuum energy goes to zero.

3.3 Recovering the first order complex deformation

Let us see in more detail how the subdominant piece in m_α will come into the game, as an ambiguity in defining the variables with mixed SU(2) indices, i.e. b_{12}, c_{112} and c_{122}. For instance we can define

$$\eta \equiv b_{21} - b_{12} = Y_3 U_L^\alpha \mathcal{M}_\alpha = y u^\alpha m'_\alpha = \mathcal{O}(\epsilon^{-2}). \tag{3.34}$$

The ambiguity increases as $\epsilon \to 0$, but is vastly subdominant with respect to the leading behavior of b. Hence at infinity one finds back the algebraic description (2.14). This shows that, indeed, one can have a supersymmetric configuration on the mesonic branch corresponding to a D3-brane at infinity.

There is a first order complex deformation of the first del Pezzo cone, which was given by Altmann [15] (see also [9]):

$$\begin{aligned}
a_1(b_2-3\sigma) &= a_2 b_1 & a_1 b_3 &= a_2 b_2 \\
b_2(b_2-3\sigma) &= b_1 b_3 & b_2(b_2-2\sigma) &= a_1 c_3 \\
b_2(b_2-4\sigma) &= a_2 c_2 & b_1^2 &= a_1 c_1 \\
b_3^2 &= a_2 c_4 & a_1 c_2 &= b_1(b_2-\sigma) \\
a_1 c_4 &= b_2 b_3 & a_2 c_1 &= b_1(b_2-3\sigma) \\
a_2 c_3 &= (b_2-2\sigma) b_3 & b_1 c_2 &= (b_2-\sigma) c_1 \\
b_1 c_3 &= (b_2-\sigma) c_2 & b_1 c_4 &= (b_2-\sigma) c_3 \\
(b_2-2\sigma) c_3 &= b_3 c_2 & (b_2-2\sigma) c_4 &= b_3 c_3 \\
(b_2-2\sigma) c_2 &= b_3 c_1 & c_1 c_4 &= c_2 c_3 \\
c_2^2 &= c_1 c_3 & c_3^2 &= c_2 c_4
\end{aligned} \tag{3.35}$$

It is natural to ask whether there is a relation between our ambiguity parameter η and this deformation parameter σ, which we recall has to satisfy $\sigma^2 = 0$ for consistency. We note here that Altmann's deformation only affects the relations where $b_2 \equiv b_{12}$ appears. In our case, also c_2 and c_3 would likely be affected.

In order to take into account the ambiguity, we give a more general definition of the loop variables, keeping the distinction between u_α and m_α. It reads as follows

$$\begin{aligned}
a_1 &= y^2 u_1, & b_1 &= y u_1 m_1, & c_1 &= u_1 m_1^2, \\
a_2 &= y^2 u_2, & b_2 &= y u_1 m_2, & c_2 &= u_1 m_1 m_2, \\
& & b_3 &= y u_2 m_2, & c_3 &= u_1 m_2^2, \\
& & & & c_4 &= u_2 m_2^2.
\end{aligned} \tag{3.36}$$

The ambiguity will arise when terms like b_{21} appear. It is taken care of by defining

$$u_2 m_1 = u_1 m_2 + \eta'. \tag{3.37}$$

We can now use the above definitions to write how the relations (2.14) are deformed

$$\begin{aligned}
a_1(b_2+\eta) &= a_2 b_1 & a_1 b_3 &= a_2 b_2 & b_2(b_2+\eta) &= b_1 b_3 & b_2^2 &= a_1 c_3 \\
b_2(b_2+\eta) &= a_2 c_2 & b_1^2 &= a_1 c_1 & b_3^2 &= a_2 c_4 & a_1 c_2 &= b_1 b_2 \\
a_1 c_4 &= b_2 b_3 & a_2 c_1 &= b_1(b_2+\eta) & a_2 c_3 &= b_2 b_3 & b_1 c_2 &= b_2 c_1 \\
b_1 c_3 &= b_2 c_2 & b_1 c_4 &= (b_2+\eta) c_3 & (b_2+\eta) c_3 &= b_3 c_2 & b_2 c_4 &= b_3 c_3 \\
(b_2+\eta) c_2 &= b_3 c_1 & c_1 c_4 &= c_2(c_3+\tilde{\eta}) & c_2^2 &= c_1 c_3 & c_3(c_3+\tilde{\eta}) &= c_2 c_4
\end{aligned} \tag{3.38}$$

where we have defined $\tilde{\eta} = m_2\eta'$. As we had anticipated, some relations involving only cs are also deformed, in contradistinction with (3.35). However, it is possible to shift the c_2 and c_3 variables in such a way that the last three relations above are not deformed. This is realized by

$$c_2 = c_2' - \frac{1}{3}m_1\eta', \qquad c_3 = c_3' - \frac{2}{3}m_2\eta'. \tag{3.39}$$

As with Altmann's deformation, we are here only considering the first order deformations, that is we formally impose $\eta'^2 = 0$.

Using now the shifted variables above, we can rewrite all the relations (3.38). For instance, take the upper right one, $b_2^2 = a_1c_3$. In terms of the shifted variables it reads

$$b_2\left(b_2 + \frac{2}{3}\eta\right) = a_1c_3', \tag{3.40}$$

so that after identifying

$$\eta \equiv -3\sigma, \tag{3.41}$$

we recover exactly the right deformed equation as in (3.35). Performing the same shifts in the other relations we recover exactly, including all numerical factors, the deformations found by Altmann.

We thus see that regular D3-branes probing the geometry not only know about the singular cone, but also about its first order complex deformation. It is because the deformation is only supersymmetric at first order that the branes are pushed to infinity on the mesonic branch.

We note here that a supergravity approach to deforming the cone over dP_1 in the gauge/gravity context has appeared in [20]. It is not immediately clear whether the first order deformation discussed there exactly maps to Altmann's, described by σ above. It would be very interesting to understand how the deformation of [20] translates into the equations defining the CY cone.

3.4 Runaway in the $N \neq M$ cases

Having understood in detail the previous case, we can work out in all generality the case $N \neq M$ along very similar lines. As in the conifold case treated in appendix A, we assume that the effect of the quantum dynamics is to produce an effective ADS-like term in the superpotential. This is really the ADS superpotential generated by instanton or gaugino condensation effects when $N < M$. For $N > M$, the term can be thought of as the result of integrating out the magnetic quarks when the mesons have (large) VEVs. In any event, the form of the potential is completely fixed, up to a numerical factor, by the symmetries of the problem. Hence, we write

$$W = h\mathrm{Tr}(\epsilon^{\alpha\beta}Y_3U_{L\alpha}\mathcal{M}_\beta - \epsilon^{\alpha\beta}V_\alpha\mathcal{N}_\beta + \epsilon^{\alpha\beta}V_\alpha U_{L\beta}Y_1) + (M-N)\left(\frac{\Lambda_2^{N+7M}}{\det\widetilde{\mathcal{M}}}\right)^{\frac{1}{M-N}}. \tag{3.42}$$

The F-term equations derived from the superpotential above will be much similar as before. The eqs. (3.6)–(3.9) remain unchanged, while the eqs. (3.4)–(3.5) become

$$\epsilon^{\alpha\beta} Y_3 U_{L\alpha} = \left(\frac{\Lambda_2^{N+7M}}{\det \widetilde{\mathcal{M}}}\right)^{\frac{1}{M-N}} \frac{1}{\det \widetilde{\mathcal{M}}} \frac{\partial \det \widetilde{\mathcal{M}}}{\partial \mathcal{M}_\beta}, \tag{3.43}$$

$$\epsilon^{\alpha\beta} V_\alpha = -\left(\frac{\Lambda_2^{N+7M}}{\det \widetilde{\mathcal{M}}}\right)^{\frac{1}{M-N}} \frac{1}{\det \widetilde{\mathcal{M}}} \frac{\partial \det \widetilde{\mathcal{M}}}{\partial \mathcal{N}_\beta}. \tag{3.44}$$

Of course, the F-terms involving the baryons are no longer present. In the following it will be convenient to introduce the shorthand

$$\mathcal{L} \equiv \left(\frac{\Lambda_2^{N+7M}}{\det \widetilde{\mathcal{M}}}\right)^{\frac{1}{M-N}}. \tag{3.45}$$

We can now attempt to solve the F-terms using an ansatz exactly similar to (3.12)–(3.14), except that now $m_\alpha, n_\alpha, y_3, v_\alpha, y_1$ and u_α are $N \times N$ diagonal matrices, while $\epsilon_\alpha, \delta_\alpha, w_\alpha$ and x_α are $M \times M$ diagonal matrices.

We can further simplify the problem by taking all the matrices to be proportional to the identity. Of course, as far as the $N \times N$ matrices are concerned, we really want ultimately all the eigenvalues to be distinct, but the scalings discussed below will not change.

Thus, introducing the ansatz above in the F-term equations, we will obtain simplified equations which consist of (3.16)–(3.21) together with (up to an N, M-dependent sign)

$$y_3 u_\alpha = \mathcal{L} \frac{1}{m_\beta n^\beta} n_\alpha, \tag{3.46}$$

$$v_\alpha = \mathcal{L} \frac{1}{m_\beta n^\beta} m_\alpha, \tag{3.47}$$

$$w_\alpha = -\mathcal{L} \frac{1}{\epsilon_\beta \delta^\beta} \delta_\alpha, \tag{3.48}$$

$$x_\alpha = \mathcal{L} \frac{1}{\epsilon_\beta \delta^\beta} \epsilon_\alpha. \tag{3.49}$$

We then again take $\epsilon_1 = \delta_2 = 0$ and $\epsilon_2 = \delta_1 = \epsilon$. This implies $x_1 = w_2 = 0$. We further simplify and solve more F-terms by taking $y_1 = y_3 = y$, $v_\alpha = m_\alpha$ and $n_\alpha = y u_\alpha$. Then the F-terms (3.46)–(3.47) are solved by

$$\mathcal{L} = y m_\alpha u^\alpha. \tag{3.50}$$

We eventually recover as before that all the F-terms are satisfied if we take $\epsilon \to 0$ together with

$$m_\alpha u^\alpha = \mathcal{O}(\epsilon). \tag{3.51}$$

This again implies that there is a subleading component in m_α, and we made the (arbitrary) choice of scaling all the non-vanishing F-terms to zero in the same way.

The scaling (to infinity) of y is determined in the following way. Eq. (3.50) becomes now $\mathcal{L} = y\mathcal{O}(\epsilon)$. However $\mathcal{L}$ is an expression involving y and ϵ. Indeed, up to a sign

$$\det \widetilde{\mathcal{M}} = y^N (m_\alpha u^\alpha)^N \epsilon^{2M} \sim y^N \epsilon^{N+2M}. \tag{3.52}$$

It is then easy to see that $\mathcal{L} \sim y\epsilon$ implies

$$y = \mathcal{O}(\epsilon^{-3}). \tag{3.53}$$

We can then also take u_α to scale in the same way. Thus, all the scalings are exactly the same as in the previous simple case of $N = M = 1$, and we are led to the same conclusions regarding the asymptotic behavior of the loop variables and their ambiguities.

In particular, the ambiguity parameter η which is eventually equated to the first order deformation parameter, is directly proportional to $\mathcal{L}$, which in turn is proportional to the gaugino condensate S for the second node (strictly speaking, in the case $N < M$).

Note that with the scalings above, $\mathcal{L} = \mathcal{O}(\epsilon^{-2})$. Thus it scales like the Lagrange multiplier in the previous case, and actually one can show that it is indeed formally replaced by the Lagrange multiplier when $N = M$ in all generality. This scaling also implies that the gaugino condensate grows unboundedly along the runaway direction.

As a last curiosity, we can compute how the determinant scales

$$\det \widetilde{\mathcal{M}} \sim \epsilon^{2(M-N)}. \tag{3.54}$$

Quite intuitively, the determinant goes to zero when it has a predominance of zero eigenvalues (in the $N < M$ case) while it goes to infinity when there are more eigenvalues scaling to infinity (when $N > M$).

The global picture after this analysis is the following. If we start with a number of regular branes N much larger than the number of fractional ones M, we see that we always have the option of trying to explore the mesonic moduli space, which is represented by the regular branes wandering around the geometry. However, this mesonic moduli space, for any value of N, is actually lifted because of the presence of the fractional branes, and the regular branes are pushed to infinity (as anticipated in [10] for the case of one probe regular D3-brane), where they explore a geometry very close to the singular complex cone over dP_1. This runaway behavior seems to have the same strength irrespective of the relative numbers N and M. However we recall that there was some freedom to choose the scaling of the variables, so that this dynamical issue is not settled at this level of the analysis.

At any given N, one has however also the option of exploring the "baryonic" branch of the moduli space, which for $N > M$ amounts to performing a Seiberg duality on the second node. This restitutes the same quiver but with ranks shifted according to $N \to N - M$. Then at any further step one finds again the alternative between going on a runaway mesonic branch or performing a further step. At the last step, we either end up with a runaway mesonic branch at $N < M$, or if $N = M$, we have a last option of going toward a baryonic branch, which however is itself runaway (albeit differently) as was already known. Note that in this last case, we do not have a pictorial way of representing the runaway as some branes being pushed to infinity. It would be nice to understand this better.

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4. Discussion

It was suggested in [9] that there was a one to one correspondence between CY singularities with obstructed deformations and quiver gauge theories with runaway supersymmetry breaking in the deep IR. In the present paper we have shown that, in the example of the dP_1 geometry, the relation is even more precise. At higher steps of the cascade, there is also a runaway behavior along the mesonic branches, which reproduce exactly the first order deformation of the geometry. Given the genericity of runaway behavior in quiver gauge theories (see e.g. [21, 22]), we expect that obstructed deformations can be reproduced similarly in generic toric singularities, even when non-obstructed deformations are possible. Indeed, in appendix B we show that this is true for the dP_2 singularity.

It is nice to see that the correspondence between quiver gauge theories and D-branes at singularities remains valid beyond issues pertaining to supersymmetric vacua. This was also argued to hold for theories displaying metastable vacua (see [23] for examples where metastability can be argued for on both sides of the correspondence). The situation discussed here is qualitatively different and can thus be considered as further evidence.

On the mesonic branches, the runaway is naturally interpreted as D3-branes being pushed to infinity. Unfortunately we do not have as nice an interpretation of the runaway along the baryonic branch. As argued in [10], it could be related to the blowing up of a closed string modulus, namely a dynamical FI term. This blown-up background must somehow have an unbalanced D3-charge/tension ratio, so that the additional regular D3 branes feel a repulsive force in its presence. Based on the findings presented here, it is tempting to speculate that if a supergravity dual of the baryonic branch runaway exists, a crucial role in its construction should be played by a (non-supersymmetric) completion of the first order deformation of the dP_1 cone. Moreover, there should be a signal of a diverging gluino condensate. Presumably, a singularity is impossible to avoid, at least in a static solution.

Having many runaway directions might eventually be interesting in cosmology, which is the only framework to make sense of such theories with no vacuum. In particular, there can be different regions of the universe where the runaway is taking place along a different direction. There would then be domain walls between those regions (possibly bubble walls if the runaway is faster in some specific direction). Those will be NS5-branes wrapped on the topological S^3 of the base. This can be seen using the same arguments as in [24, 19] and noticing that the domain walls would interpolate between regions with a different number of D3-branes. Note that for the domain wall tension to be non vanishing, the 3-cycle wrapped by the NS5-branes must be of finite size. This is non trivial in the absence of a consistent deformation (i.e. a blown up 3-cycle). We are left to suppose that the 3-form flux sourced by the fractional branes somehow prevents the collapse of the NS5-brane worldvolume, possibly due to dynamics which is necessarily time-dependent.

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A. Classical and quantum moduli spaces of the conifold theory

We consider here the warm-up example of the quiver gauge theory resulting from N D3-branes at a conifold singularity, with the addition of M fractional branes. The gauge groups are $\mathrm{SU}(N) \times \mathrm{SU}(N+M)$, and there are two pairs of bifundamentals $A^i_{\alpha a}$ and $B^a_{\alpha i}$ where $\alpha = 1, 2$ and i and a are indices in the first and second gauge group respectively. We aim here at reproducing in a compact way the results of [8].

A.1 Classical analysis

The classical tree level superpotential is

$$W = h A^i_{\alpha a} B^a_{\beta j} A^j_{\gamma b} B^b_{\delta i} \epsilon^{\alpha\gamma} \epsilon^{\beta\delta}. \tag{A.1}$$

From it we derive the F-terms

$$A^i_{\alpha a} B^a_{\beta j} A^j_{\gamma b} \epsilon^{\alpha\gamma} = 0, \qquad B^a_{\beta j} A^j_{\gamma b} B^b_{\delta i} \epsilon^{\beta\delta} = 0. \tag{A.2}$$

The above F-terms can be written in a more interesting way if contracted so as to form gauge invariants of the first or the second gauge group. We call $M^i_{\alpha\beta j} = A^i_{\alpha a} B^a_{\beta j}$ and $\tilde{M}^a_{\alpha\beta b} = B^a_{\alpha i} A^i_{\beta b}$, and we obtain, in matrix notation

$$M_{\alpha\beta} M_{\gamma\delta} \epsilon^{\alpha\gamma} = 0 = M_{\alpha\beta} M_{\gamma\delta} \epsilon^{\beta\delta}, \qquad \tilde{M}_{\alpha\beta} \tilde{M}_{\gamma\delta} \epsilon^{\alpha\gamma} = 0 = \tilde{M}_{\alpha\beta} \tilde{M}_{\gamma\delta} \epsilon^{\beta\delta}. \tag{A.3}$$

The above equations read, component by component

$$\begin{aligned}
&M_{11}M_{21} = M_{21}M_{11}, && M_{11}M_{22} = M_{21}M_{12}, \\
&M_{12}M_{21} = M_{22}M_{11}, && M_{12}M_{22} = M_{22}M_{12}, \\
&M_{11}M_{12} = M_{12}M_{11}, && M_{11}M_{22} = M_{12}M_{21}, \\
&M_{21}M_{12} = M_{22}M_{11}, && M_{21}M_{22} = M_{22}M_{21},
\end{aligned} \tag{A.4}$$

or, in a more compact way

$$[M_{\alpha\beta}, M_{\gamma\delta}] = 0, \qquad M_{11}M_{22} - M_{12}M_{21} = 0. \tag{A.5}$$

The same holds for the matrices $\tilde{M}_{\alpha\beta}$. As a consequence, using gauge transformations of $\mathrm{SU}(N)$ and $\mathrm{SU}(N+M)$ respectively, one can diagonalize both sets of 4 commuting matrices $M_{\alpha\beta}$ and $\tilde{M}_{\alpha\beta}$. Note that the latter matrices are not of maximal rank $N+M$, but rather only of rank N. Hence, they will have generically M vanishing eigenvalues.

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Eigenvalue by eigenvalue, we have that

$$m_{11}^{(i)} m_{22}^{(i)} - m_{12}^{(i)} m_{21}^{(i)} = 0. \tag{A.6}$$

These are N copies of the equation defining the conifold singularity, $xy = uv$. If we define

$$\mathcal{M} \equiv \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \tag{A.7}$$

we immediately see that

$$\det \mathcal{M} = 0. \tag{A.8}$$

As for $\tilde{M}_{\alpha\beta}$, we can take the first N entries of, say, $\tilde{M}_{11}$ to be non vanishing. Then the relations similar to (A.6) are most generically satisfied by non vanishing eignevalues when also the other 3 matrices have non vanishing first N entries.

At this stage, let us go back to the F-term conditions written in terms of the elementary fields. For instance, we have the following expression (even before imposing the F-terms)

$$M_{11} A_1 = A_1 B_1 A_1 = A_1 \tilde{M}_{11}. \tag{A.9}$$

For a generic matrix A_1 and M_{11}, $\tilde{M}_{11}$ as above, we have

$$m_{11}^{(i)} A_{1a}^{i} = A_{1a}^{i} \tilde{m}_{11}^{(a)}. \tag{A.10}$$

As we need some components of A_1 to be non zero (since after all M_{11} and $\tilde{M}_{11}$ are built from it), we see that we must have $m_{11}^{(i)} = \tilde{m}_{11}^{(i)}$ and $A_{1a}^{i} = 0$ for $i \neq a$.

Now, using the F-terms we also obtain that

$$M_{12} A_1 = A_1 \tilde{M}_{21}, \quad M_{21} A_1 = A_1 \tilde{M}_{12}, \quad M_{22} A_1 = A_1 \tilde{M}_{22}, \tag{A.11}$$

so that

$$m_{12}^{(i)} = \tilde{m}_{21}^{(i)}, \quad m_{21}^{(i)} = \tilde{m}_{12}^{(i)}, \quad m_{22}^{(i)} = \tilde{m}_{22}^{(i)}. \tag{A.12}$$

Finally, using relations based on $A_2 B_2 A_2$, $B_1 A_1 B_1$ and $B_2 A_2 B_2$ we obtain that all elementary fields A_α, B_α can be taken to be diagonal in their upper/left $N \times N$ piece.

Note that we did not use until now information coming from requiring D-flatness. The only constraint left is

$$|a_1^{(i)}|^2 + |a_2^{(i)}|^2 = |b_1^{(i)}|^2 + |b_2^{(i)}|^2. \tag{A.13}$$

A.2 Quantum analysis

We now want to take into account quantum corrections to the above story. We do this by considering that the node with largest rank $\mathrm{SU}(N+M)$ goes to strong coupling first. Then, its dynamics should be effectively described by gauge invariants, which in this case are the $M_{\alpha\beta}$ matrices with indices in the first gauge group, which will be considered as classical in these considerations.

The quantum corrections in a region of the moduli space where the mesons $M_{\alpha\beta}$ have large enough VEVs (i.e. the so-called mesonic branch) are captured by adding an ADS-like

superpotential. It can be seen to arise in the Seiberg dual picture from integrating out the dual magnetic quarks which are massive because of the mesonic VEVs. We thus write

$$W_{\text{eff}} = hM_{\alpha\beta}M_{\gamma\delta}\epsilon^{\alpha\gamma}\epsilon^{\beta\delta} - (N-M)\left(\frac{\Lambda^{N+3M}}{\det\mathcal{M}}\right)^{\frac{1}{M-N}}, \tag{A.14}$$

where Λ is the dynamical scale of the strongly coupled node. Note that for $N < M$, this is really an ADS superpotential. For $N > M$, the determinant has actually a positive power. The case $N = M$ is analyzed below in more detail.

Extremizing with respect to $M_{\alpha\beta}$, we obtain

$$h\begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix} = \left(\frac{\Lambda^{N+3M}}{\det\mathcal{M}}\right)^{\frac{1}{M-N}}\mathcal{M}^{-1}. \tag{A.15}$$

Multiplying by $\mathcal{M}$ these equations from the right and from the left, we obtain matrix equations which imply

$$[M_{\alpha\beta}, M_{\gamma\delta}] = 0, \qquad M_{11}M_{22} - M_{12}M_{21} = \frac{1}{h}\left(\frac{\Lambda^{N+3M}}{\det\mathcal{M}}\right)^{\frac{1}{M-N}}. \tag{A.16}$$

As before, the matrices $M_{\alpha\beta}$ can all be simultaneously diagonalized, and their eigenvalues must satisfy

$$m_{11}^{(i)}m_{22}^{(i)} - m_{12}^{(i)}m_{21}^{(i)} = \frac{1}{h}\left(\frac{\Lambda^{N+3M}}{\prod_j(m_{11}^{(j)}m_{22}^{(j)} - m_{12}^{(j)}m_{21}^{(j)})}\right)^{\frac{1}{M-N}}. \tag{A.17}$$

Taking the product of all the N such equations, we eventually obtain

$$\det\mathcal{M} = \prod_i(m_{11}^{(i)}m_{22}^{(i)} - m_{12}^{(i)}m_{21}^{(i)}) = \left(h^{N-M}\Lambda^{N+3M}\right)^{\frac{N}{M}}. \tag{A.18}$$

Reinserting in (A.17), we obtain

$$m_{11}^{(i)}m_{22}^{(i)} - m_{12}^{(i)}m_{21}^{(i)} = \left(h^{N-M}\Lambda^{N+3M}\right)^{\frac{1}{M}} = \Lambda^4(h\Lambda)^{\frac{N-M}{M}}. \tag{A.19}$$

We thus see that we have N copies of the deformed conifold, defined by $xy - uv = \epsilon$. Note that the deformation parameter is parametrically smaller as N is increased, since h can be taken to be of the order of the inverse string scale.[6]

Thus we see that when fractional branes are present, the moduli space probed by regular branes is smoothened to the deformed conifold because of quantum effects.

When there are no fractional branes, $M = 0$, the equations (A.17) can be satisfied only if $\det\mathcal{M} = 0$, which implies eventually (A.6), i.e. the moduli space remains the classical, singular conifold.

Note that there is a subtle point in this specific case. The F-terms (A.15) would seem to imply that the mesons actually have to vanish. This is clearly a too strong constraint. Hence, requiring F-flatness of the effective superpotential in this case seems to be misleading. Possibly, this is due to the strictly conformal nature of the quiver gauge theory, which prevents us to consider one node as strongly coupled and the other as classical.

[6]The string scale is effectively warped down from its true value in the deep UV. This can be seen as an effect of the cascading RG flow.

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A.3 When baryonic branches are present

We are left to analyze the case $N = M$, which we take to be a case study of the case $N = kM$ where baryonic operators are allowed. At the classical level, we can write two more gauge invariants of the second node, which turn out to be gauge invariant also with respect to the first one. Indeed, node two has $N_f = N_c$ and we can write

$$\begin{aligned}\mathcal{B} &= \epsilon_{i_1\dots i_{2M}}\epsilon^{a_1\dots a_M}\epsilon^{b_1\dots b_M}A^{i_1}_{1a_1}\dots A^{i_M}_{1a_M}A^{i_{M+1}}_{2b_1}\dots A^{i_{2M}}_{2b_M},\\ \tilde{\mathcal{B}} &= \epsilon^{i_1\dots i_{2M}}\epsilon_{a_1\dots a_M}\epsilon_{b_1\dots b_M}B^{a_1}_{1i_1}\dots B^{a_M}_{1i_M}B^{b_1}_{2i_{M+1}}\dots B^{b_M}_{2i_{2M}}.\end{aligned} \tag{A.20}$$

Still at the classical level, we see that we can form new gauge invariants from the F-terms such as

$$M_{\alpha\beta}\mathcal{B} = 0, \qquad M_{\alpha\beta}\tilde{\mathcal{B}} = 0. \tag{A.21}$$

It implies that we can turn on either the baryonic VEVs or the mesonic ones, but not both at the same time. Moreover, the classical constraint $\det\mathcal{M} = \mathcal{B}\tilde{\mathcal{B}}$ implies that $\det\mathcal{M} = 0$ on the mesonic branch (this was already derived above) and that $\mathcal{B}\tilde{\mathcal{B}} = 0$ on the baryonic branch, which is thus separated in two components.

The mesonic branch is derived exactly as before, so that the complete moduli space in this case is the sum of the symmetric product of M copies of the conifold (parametrized by $M_{\alpha\beta}$) and two complex lines (parametrized by $\mathcal{B}$ and $\tilde{\mathcal{B}}$). All three components of the moduli space meet at the origin of each branch.

At the quantum level, the effective strongly coupled dynamics of the second node induces a deformation of its classical moduli space. Such a deformation is encoded in the following effective superpotential which includes a Lagrange multiplier L

$$W_{\text{eff}} = hM_{\alpha\beta}M_{\gamma\delta}\epsilon^{\alpha\gamma}\epsilon^{\beta\delta} + L(\det\mathcal{M} - \mathcal{B}\tilde{\mathcal{B}} - \Lambda^{4M}). \tag{A.22}$$

The F-terms are the following

$$h\begin{pmatrix} M_{22} & -M_{12}\\ -M_{21} & M_{11}\end{pmatrix} = L(\det\mathcal{M})\mathcal{M}^{-1}, \tag{A.23}$$

$$L\mathcal{B} = 0 = L\tilde{\mathcal{B}}, \tag{A.24}$$

together with the constraint

$$\det\mathcal{M} - \mathcal{B}\tilde{\mathcal{B}} = \Lambda^{4M}. \tag{A.25}$$

It is clear that we have a baryonic branch where the $\mathcal{B}, \tilde{\mathcal{B}} \neq 0$. This implies that $L = 0$ and in turn $M_{\alpha\beta} = 0$. The two classical baryonic branches have merged into one $\mathcal{B}\tilde{\mathcal{B}} = -\Lambda^{4M}$.

If we want the mesons to be non vanishing, we need to have $L \neq 0$, which forces the baryons to vanish. Then we automatically get $\det\mathcal{M} = \Lambda^{4M}$, the $M_{\alpha\beta}$ commute and eigenvalue by eigenvalue we have

$$m^{(i)}_{11}m^{(i)}_{22} - m^{(i)}_{12}m^{(i)}_{21} = \Lambda^4, \tag{A.26}$$

which also sets $L = h\Lambda^{4-4M}$.

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Thus we see that at the quantum level, we still have two components, one being the one complex dimensional baryonic branch and the other being the symmetric product of M copies of the deformed conifold. This time the two branches are both completely smooth[7] and do not touch.

The full moduli space of the theory for a given N, can be derived component by component in the way described here, reproducing the results of [8]. The first component is described by N copies of the deformed conifold, corresponding to the mesonic branch. However if the mesons are not given VEVs, one can Seiberg dualize the strongly coupled node and reach a theory where effectively N is replaced by $N - M$. At every step in this cascade of dualities there is a component of the moduli space which will be described by $N - kM$ copies of the deformed conifold. If N is a multiple of M, we end up with a smooth baryonic branch, while if it is not the smallest component of the moduli space will still be a mesonic branch corresponding to $N - k_{\max}M < M$ D3 branes on the deformed conifold.

B. Quantum corrections to the dP_2 moduli space

B.1 The space of complex deformations for dP_2

The complex cone over dP_2 admits two different kinds of fractional branes, according to the classification of [10]. One is a deformation brane, corresponding to a complex structure deformation of the cone. The corresponding gauge theory was studied in detail in [25], where it was shown that the deformed chiral algebra encodes precisely the complex deformation computed according to Altmann's rules [15]. The second fractional brane allowed by the geometry is a so called supersymmetry breaking (SB) brane, which corresponds in this case to an obstructed complex deformation of the geometry.

This section follows closely the work of [25], to which we refer for further details on the application of Altmann's techniques. Using the usual toric geometry techniques, one can describe the dP_2 cone as an affine variety in $\mathbb{C}^8$. Let $(a_1, a_2, b_1, b_2, b_3, c_1, c_2, d) \in \mathbb{C}^8$ be the complex coordinates corresponding to the generators of the dual toric cone $\sigma^\vee$. There are 14 relations amongst these:

$$\begin{array}{lll} b_2^2 = b_1 b_3, & b_2^2 = a_1 c_2, & b_2^2 = c_1 a_2, \\ c_1^2 = b_1 d\ , & c_2^2 = b_3 d\ , & \\ b_1 a_2 = b_2 a_1, & c_1 b_2 = c_2 b_1, & b_2 a_2 = b_3 a_1, \\ c_1 b_3 = c_2 b_2, & b_1 b_2 = c_1 a_1, & b_2 b_3 = c_2 a_2, \\ c_1 c_2 = b_2 d\ , & c_1 b_2 = a_1 d\ , & c_2 b_2 = a_2 d\ . \end{array} \tag{B.1}$$

The Minkowski cone (that is the cone of Minkowski summands of the toric diagram) is given by

$$\begin{array}{lll} (t_1, \ldots, t_5) & \text{s.th.} & P_1(t) = t_1 - t_2 - t_3 + t_5 = 0, \\ & & P_2(t) = t_1 + t_2 - t_4 - t_5 = 0, \end{array} \tag{B.2}$$

[7] Except of course for singularities due to the symmetric product orbifold action.

and $t_i \geq 0$. We parametrise it by

$$t_1 = t, \quad t_2 = t - s_1, \quad t_3 = t - s_2, \quad t_4 = t + s_2, \quad t_5 = t - s_1 - s_2. \tag{B.3}$$

The space of deformations is given by imposing the further constraints $P_{1,2}(t^2) = 0$. We thus have s_1 and s_2 subject to the quadratic constraints

$$s_1 s_2 = 0, \qquad s_2^2 = 0. \tag{B.4}$$

Clearly, one solution is $s_2 = 0$. This corresponds to the deformation brane case studied in [25]. But we are interested here in the case $s_1 = 0$, $s_2^2 = 0$. Then, s_2 corresponds to a first order deformation obstructed at second order (corresponding to the SB branes), similarly to the dP_1 case. Running Altmann's algorithm, which replaces the coordinate b_2 by the five new variables t_i, we can show that the 14 relations (B.1) become

$$\begin{array}{rrr}
t_1^2 = b_1 b_3, & t_2 t_4 = a_1 c_2, & t_1 t_2 = c_1 a_2, \\
c_1^2 = b_1 d, & c_2^2 = b_3 d, & \\
b_1 a_2 = \frac{t_1^2}{t_4} a_1, & c_1 t_1 = c_2 b_1, & t_4 a_2 = b_3 a_1, \\
c_1 b_3 = c_2 t_1 & b_1 \frac{t_2 t_4}{t_1} = c_1 a_1, & t_2 b_3 = c_2 a_2, \\
c_1 c_2 = t_1 d, & c_1 \frac{t_2 t_4}{t_1} = a_1 d, & c_2 t_2 = a_2 d.
\end{array} \tag{B.5}$$

Restricting to the case $s_1 = 0$, $s_2^2 = 0$, we see that $t_1 = t_2$, $t_3 = t_5$, and (renaming $t = b_2$, $s_2 = \sigma$) we eventually find

$$\begin{array}{rrr}
b_2^2 = b_1 b_3, & b_2(b_2 + \sigma) = a_1 c_2, & b_2^2 = c_1 a_2, \\
c_1^2 = b_1 d, & c_2^2 = b_3 d, & \\
b_1 a_2 = (b_2 - \sigma) a_1, & c_1 b_2 = c_2 b_1, & (b_2 + \sigma) a_2 = b_3 a_1, \\
c_1 b_3 = c_2 b_2 & b_1(b_2 + \sigma) = c_1 a_1, & b_2 b_3 = c_2 a_2, \\
c_1 c_2 = b_2 d, & c_1(b_2 + \sigma) = a_1 d, & c_2 b_2 = a_2 d.
\end{array} \tag{B.6}$$

Notice that indeed, for consistency, it implies that we must have $\sigma^2 = 0$.

B.2 Classical moduli space

When N D3-branes, M SB branes and P deformation branes are present on the dP_2 cone, the corresponding gauge theory has an $\mathrm{SU}(N+M+P) \times \mathrm{SU}(N+2M) \times \mathrm{SU}(N+M) \times \mathrm{SU}(N) \times \mathrm{SU}(N+P)$ gauge group. We study here the $P = 0$, $N = M = 1$ case. The field content can be read from the quiver in figure 3.

The tree level superpotential is given by

$$W_{\text{tree}} = W'YA - XYV' - ACU' + XZU'V + BCUV' - W'ZUVB, \tag{B.7}$$

where the trace is implied.

Using the F-conditions, there is a minimal set of 8 loops of the quiver that generate all mesonic gauge invariants, the chiral primaries. These are given by

$$\begin{array}{lll}
a_1 = XYV, & b_1 = BCU'V, & c_1 = BCU'V', \\
a_2 = XZUV, & b_2 = XZU'V, & c_2 = XZU'V', \\
b_3 = XZUV', & d = BW'ZU'V', &
\end{array} \tag{B.8}$$

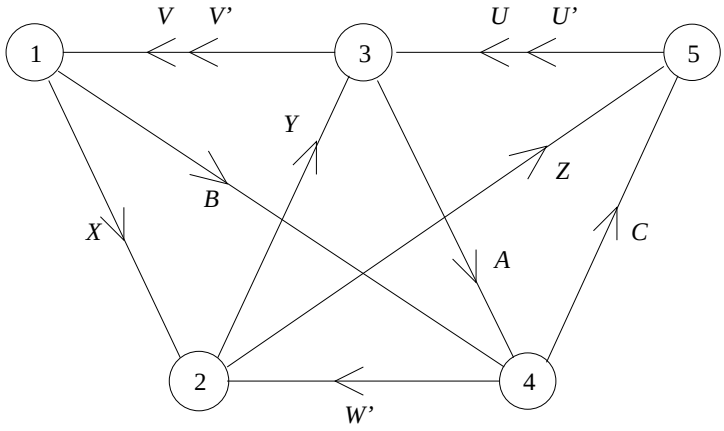

Figure 3: The dP_2 quiver.

where we chose to base all loops at the first node. Note that there is a grading in term of the number of primed fields. There are 14 relations amongst these 8 fields, defining the complex cone over dP_2 as in (B.1).

B.3 Runaway on the mesonic branch

We will consider the quantum corrections from the second node to be dominant. It has gauge group SU(3) and $N_f = N_c$. As for dP_1, we must consider its mesons as effective fields,

$$M_1 = XY, \qquad M_2 = XZ, \tag{B.9}$$

$$M_3 = W'Y, \qquad M_4 = W'Z. \tag{B.10}$$

Let us also define the (3×3) meson matrix,

$$\widetilde{\mathcal{M}} \equiv \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}. \tag{B.11}$$

The quantum contribution to the superpotential is of course

$$W_{\rm qu} = L(\det \widetilde{\mathcal{M}} - \bar{\mathcal{B}}\bar{\mathcal{B}} - \Lambda_2^6). \tag{B.12}$$

We want to analyse the behavior of the mesonic branch, so we will impose the above constraint as $\det \widetilde{\mathcal{M}} = \Lambda_2^6$. Similarly to the dP_1 case, we can take all bifundamental fields to be upper-left diagonal:

$$\begin{aligned}
M_1 &= \begin{pmatrix} m_1 & 0 \\ 0 & \epsilon \end{pmatrix}, \qquad & M_2 &= \begin{pmatrix} m_2 \\ 0 \end{pmatrix}, \qquad & M_3 &= \begin{pmatrix} m_3 & 0 \end{pmatrix}, \qquad & M_4 &= m_4, \\
A &= \begin{pmatrix} a \\ 0 \end{pmatrix}, & B &= \begin{pmatrix} b \\ 0 \end{pmatrix}, & C &= c, & & \\
U &= \begin{pmatrix} u & 0 \end{pmatrix}, & V &= \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}, & U' &= \begin{pmatrix} u' & 0 \end{pmatrix}, & V' &= \begin{pmatrix} v' & 0 \\ 0 & \tilde{v}' \end{pmatrix}.
\end{aligned} \tag{B.13}$$

The non-upper-left elements ϵ and $\tilde{v}'$ are of course there to reconcile the rank condition and the constraint.

The constraint is thus

$$\det \widetilde{\mathcal{M}} = \epsilon(m_1 m_4 - m_2 m_3) = \Lambda_2^6. \tag{B.14}$$

With the parametrisation above, the VEVs of the bifundamentals are c-numbers, and the superpotential is given by

$$\begin{aligned} W \;=\; & m_3 a - m_1 v' - \epsilon \tilde{v}' - acu' + m_2 u' v + bcuv' - m_4 uvb \\ & + L(\epsilon(m_1 m_4 - m_2 m_3) - \Lambda_2^6). \end{aligned} \tag{B.15}$$

The F-terms are

$$F_a = m_3 - cu' = 0, \tag{B.16}$$
$$F_b = cuv' - m_4 uv = 0, \tag{B.17}$$
$$F_c = -u'a + uv'b = 0, \tag{B.18}$$
$$F_u = v'bc - vbm_4 = 0, \tag{B.19}$$
$$F_v = m_2 u' - bm_4 u = 0, \tag{B.20}$$
$$F_{u'} = -ac + vm_2 = 0, \tag{B.21}$$
$$F_{v'} = -m_1 + bcu = 0, \tag{B.22}$$
$$F_{\tilde{v}'} = -\epsilon = 0, \tag{B.23}$$
$$F_\epsilon = -\tilde{v}' + L(m_1 m_4 - m_2 m_3) = 0, \tag{B.24}$$
$$F_{m_1} = -v' + L\epsilon m_4 = 0, \tag{B.25}$$
$$F_{m_2} = u'v - L\epsilon m_3 = 0, \tag{B.26}$$
$$F_{m_3} = a - L\epsilon m_2 = 0, \tag{B.27}$$
$$F_{m_1} = -uvb + L\epsilon m_1 = 0. \tag{B.28}$$

Clearly, (B.23) is incompatible with (B.14) unless $(m_1 m_4 - m_2 m_3) = \frac{\Lambda_2^6}{\epsilon} \to \infty$. We can solve for $F_{m_i} = 0$ by taking $L = \Lambda_2^6/\epsilon$ (so that the baryonic branch indeed decouples as ϵ goes to zero), and

$$\begin{aligned} m_1 &= uvb, \qquad & m_2 &= a, \\ m_3 &= u'v \qquad & m_4 &= v'. \end{aligned} \tag{B.29}$$

Then the other F-terms imply $c = v$. Moreover, (B.14) becomes

$$m_1 m_4 - m_2 m_3 = v(ubv' - au') = vF_c = \frac{\Lambda_2^6}{\epsilon}. \tag{B.30}$$

We can choose F_c to scale to zero as

$$F_c = uv'b - u'a = \mathcal{O}(\epsilon), \tag{B.31}$$

It implies that v must scale as

$$v = c = \mathcal{O}(\epsilon^{-2}). \tag{B.32}$$

One can then easily check that all F-terms are satisfied, with $F_c = F_v = 0$ that must be satisfied in the limit $\epsilon \to 0$. By taking the simplest solution $v = c$, $u = b$, $a = ub = b^2$, $u' = v'$, one can express (B.8) in term of $(b, c, u') \in \mathbb{C}^3$:

$$\begin{aligned} a_1 &= b^2c^2, \quad b_1 = bc^2u', \quad c_1 = bc(u')^2, \\ a_2 &= b^3c, \quad b_2 = b^2cu', \quad c_2 = b^2(u')^2, \\ b_3 &= b^3u', \quad d = b(u')^3, \end{aligned} \tag{B.33}$$

which implies the 14 relations (B.1). Note that all coordinates go to infinity as ϵ^{-8}.

B.4 Recovering the first order complex deformation

To conclude, let us show that the gauge theory result reproduces the first order complex deformation (B.6). The ambiguity coming from solving (B.31) can be accounted for by defining

$$v' = u' + \tilde{\eta}. \tag{B.34}$$

Let us solve for the loops as in (B.33), but taking this ambiguity into account:

$$\begin{aligned} a_1 &= b^2c^2, \quad b_1 = bc^2u', \quad c_1 = bcu'v', \\ a_2 &= b^3c, \quad b_2 = b^2cu', \quad c_2 = b^2u'v', \\ b_3 &= b^3v', \quad d = bu'(v')^2. \end{aligned} \tag{B.35}$$

Now it is an easy matter to construct the deformed relations amongst the variables of (B.35). We find

$$\begin{aligned} b_2(b_2+\eta) &= b_1b_3, & b_2(b_2+\eta) &= a_1c_2, & b_2(b_2+\eta) &= c_1a_2, \\ c_1^2 &= b_1d, & c_2(c_2+\eta') &= b_3d, & & \\ b_1a_2 &= b_2a_1, & c_1b_2 &= c_2b_1, & a_2(b_2+\eta) &= b_3a_1, \\ c_1b_3 &= c_2(b_2+\eta) & c_1a_1 &= b_1(b_2+\eta), & b_2b_3 &= c_2a_2, \\ c_1c_2 &= b_2d, & c_1(b_2+\eta) &= a_1d, & c_2(b_2+\eta) &= a_2d, \end{aligned} \tag{B.36}$$

where $\eta = b^2c\tilde{\eta}$ and $\eta' = b^2u'\tilde{\eta}$. The ambiguity parameters go to infinity as ϵ^{-1}, so they are subdominant with respect to the loop variables.

We again need to shift some variables to make contact with (B.6). An appropriate shift is

$$b_2 \to b_2 - \frac{1}{2}\eta, \qquad c_2 \to c_2 - \frac{1}{2}\eta'. \tag{B.37}$$

Of course, when plugging this into (B.36), one should consider $\eta^2 = 0$ and use the relations (B.1) when necessary. Then, identifying

$$\eta \equiv 2\sigma, \tag{B.38}$$

one recovers exactly the set of deformed equations (B.6).

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